A Green's Function Method for Treating Compton Frequency Redistribution

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Disclaimer

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Treating Compton Frequency Redistribution

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Outline

- Review the Fokker-Planck equation for Compton scattering

- Properties of the exact Compton scattering kernel

- Green's function and moments for a Liouville differential equation

- Inference of Liouville coefficients to match Compton kernel moments

- Numerical comparison of Green's function with exact scattering kernel

- Advantages of Green's function method

Fokker-Planck = Kompaneets equation

$$\frac{1}{\chi_e c}\frac{\partial n}{\partial t} = \frac{kT}{mc^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n(n+1)\right)\right]$$

$$n(\nu) \equiv \frac{c^2 J_\nu}{2h\nu^3}$$

$$x \equiv \frac{h\nu}{kT}$$

Exact scattering kernel $R(v, v')$

$$\chi_T R(v, v') = N_e \frac{v}{v'} \frac{d\sigma}{dv}$$

$$\frac{1}{\chi_T c} \frac{\partial n_v}{\partial t} = \frac{c^2}{2hv^3} \int_0^\infty dv' \left\{ -R(v', v) \frac{v}{v'} J_v \left[1 + \frac{c^2 J_{v'}}{2hv'^3} \right] + R(v, v') J_{v'} \left[1 + \frac{c^2 J_v}{2hv^3} \right] \right\}$$

$$R(v, v') = \left(\frac{v}{v'} \right)^4 \exp\left[\frac{h(v' - v)}{kT} \right] R(v', v)$$

Moments of the scattering kernel

Symmetrized scattering kernel $r(v, v')$

$$R(v, v') = \frac{v^3 \exp(-hv/kT)}{v'} r(v, v') \,, \qquad r(v', v) = r(v, v')$$

Evaluation of moments for $kT \ll mc^2$, $hv \ll mc^2$ —

$$A(v) = \int_0^\infty dv' v'^2 \exp\left(-\frac{hv'}{kT}\right)(v' - v)r(v', v) \;\approx\; \frac{kT}{mc^2}\left(4 - \frac{hv}{kT}\right)v$$

$$B(v) = \int_0^\infty dv' v'^2 \exp\left(-\frac{hv'}{kT}\right)(v' - v)^2 r(v', v) \;\approx\; \frac{2kT}{mc^2}v^2$$

Use of these moments in the Fokker-Planck method leads to Kompaneets' equation. Methods for obtaining accurate relativistic moments have been given by Prasad, *et al.*

Green's function of a Liouville differential equation

Liouville equation

$$L[y] = 0 \qquad \text{with} \qquad L[y] \equiv \frac{d}{dv}\left(p(v)\frac{dy}{dv}\right) + q(v)y$$

Green's function $G(v, v')$

$$L[y] = f(v) \qquad \rightarrow \qquad y(v) = \int_0^\infty dv' G(v, v') f(v')$$

$$\frac{\partial}{\partial v}\left(p(v)\frac{\partial G}{\partial v}\right) + q(v)G = \delta(v - v')$$

Moments of the Liouville Green's function

These integrals can be derived —

$$\int_0^\infty dv\, q(v) G(v, v') = 1$$

$$\int_0^\infty dv (v - v') q(v) G(v, v') = -\int_0^\infty dv \frac{dp}{dv} G(v, v')$$

$$\int_0^\infty dv (v - v')^2 q(v) G(v, v') = 2v' \int_0^\infty dv \frac{dp}{dv} G(v, v') - \int_0^\infty dv \frac{d}{dv}(2pv) G(v, v')$$

If $p(v) < 0$ and $|p(v)| \ll q(v)v^2$ then G is sharply peaked and

$$\int_0^\infty dv (v - v') q(v) G(v, v') \approx -\frac{1}{q(v')} \frac{dp}{dv'}$$

$$\int_0^\infty dv (v - v')^2 q(v) G(v, v') \approx -\frac{2p(v')}{q(v')}$$

The Liouville operator that matches Compton scattering moments for $kT \ll mc^2$

Comparing the Green's function moments to the small-T Compton moments suggests

$$q(v) = v^2 \exp\left(-\frac{hv}{kT}\right)$$

$$p(v) = -\frac{kT}{mc^2}v^4 \exp\left(-\frac{hv}{kT}\right)$$

and these satisfy the conditions for G to be sharply peaked provided $kT \ll mc^2$. So

$$L[G(v, v')] \equiv -\frac{kT}{mc^2}\frac{\partial}{\partial v}\left[v^4 \exp\left(-\frac{hv}{kT}\right)\frac{\partial G(v, v')}{\partial v}\right] +$$
$$v^2 \exp\left(-\frac{hv}{kT}\right) G(v, v') = \delta(v - v')$$

Expression for G in terms of special functions

The differential equation obtained for G is a variation of the confluent hypergeometric equation. With the notation

$$x \equiv \frac{h\nu}{kT}, \qquad \theta \equiv \frac{kT}{mc^2}, \qquad s = -\frac{3}{2} + \sqrt{\frac{9}{4} + \frac{1}{\theta}},$$

G is given by

$$G(\nu, \nu') = \frac{\Gamma(s)}{\Gamma(2s+4)} \frac{mc^2}{kT} \left(\frac{h}{kT}\right)^3 (xx')^s M(s, 2s+4, x_<) U(s, 2s+4, x_>)$$

M and U are Kummer (confluent hypergeometric) functions

Numerical solutions for the Green's function

The Liouville equation for G can be put into a conservative finite difference form with the method of Larsen, Levermore, Pomraning and Sanderson (1985). It reduces to a problem of finding the inverse of a tri-diagonal matrix; the matrix is almost identical to the one obtained by Larsen, *et al.*

$$H(v, v') = v^2 \exp\left(-\frac{hv}{kT}\right) \quad G(v, v') \approx \frac{v'}{v} R(v, v') = \frac{N_e}{\chi_T} \frac{d\sigma}{dv}$$

The tri-diagonal matrix U that yields H by inversion

$$H = \frac{h}{kT}U^{-1} , \qquad U = \begin{pmatrix} a_j & b_j & c_j \end{pmatrix}$$

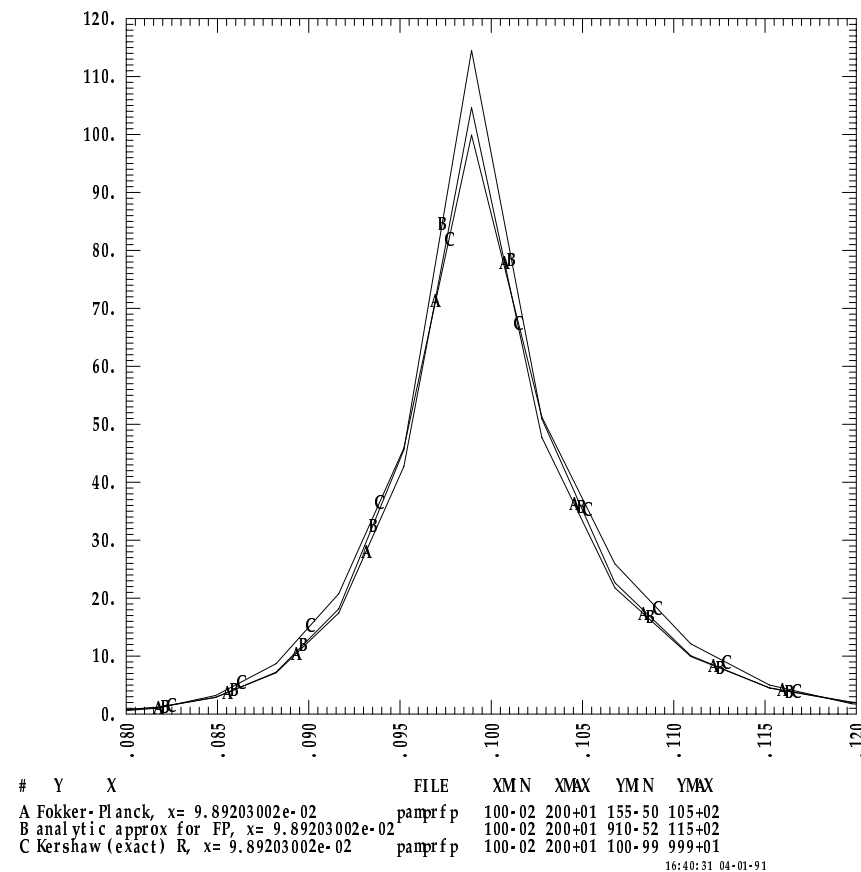$$a_j = -\frac{\theta e^{x_{j-1}}}{x_{j-1}^2} \frac{x_{j-1/2}^4}{e^{x_j} - e^{x_{j-1}}}$$

$$b_j = \frac{\theta e^{x_j}}{x_j^2}\left[\frac{x_{j-1/2}^4}{e^{x_j} - e^{x_{j-1}}} + \frac{x_{j+1/2}^4}{e^{x_{j+1}} - e^{x_j}}\right] + x_{j+1/2} - x_{j-1/2}$$

and

$$c_j = -\frac{\theta e^{x_{j+1}}}{x_{j+1}^2} \frac{x_{j+1/2}^4}{e^{x_{j+1}} - e^{x_j}}$$

Sample calculations of R

Here for $h\nu' \approx 0.1\,\text{keV}$, $kT = 1\,\text{keV}$. A is the Green's function method, B is an analytic approximation, and C is exact.
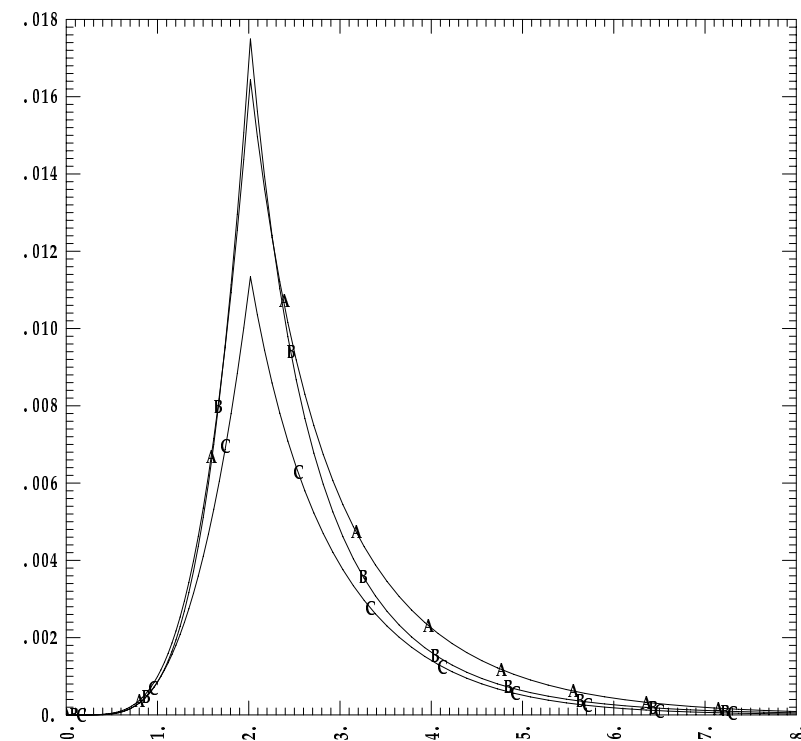
#	Y X		FILE	XMIN	XMAX	YMIN	YMAX
A	Fokker-Planck, x= 9.89203002e-02		pamprfp	100-02	200+01	155-50	105+02
B	analytic approx for FP, x= 9.89203002e-02			100-02	200+01	910-52	115+02
C	Kershaw (exact) R, x= 9.89203002e-02		pamprfp	100-02	200+01	100-99	999+01

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12

More sample calculations of R — at larger v'

For $h v' \approx 10\,\mathrm{keV}$, $kT = 1\,\mathrm{keV}$. A is the Green's function method, B is an analytic approximation, and C is exact.
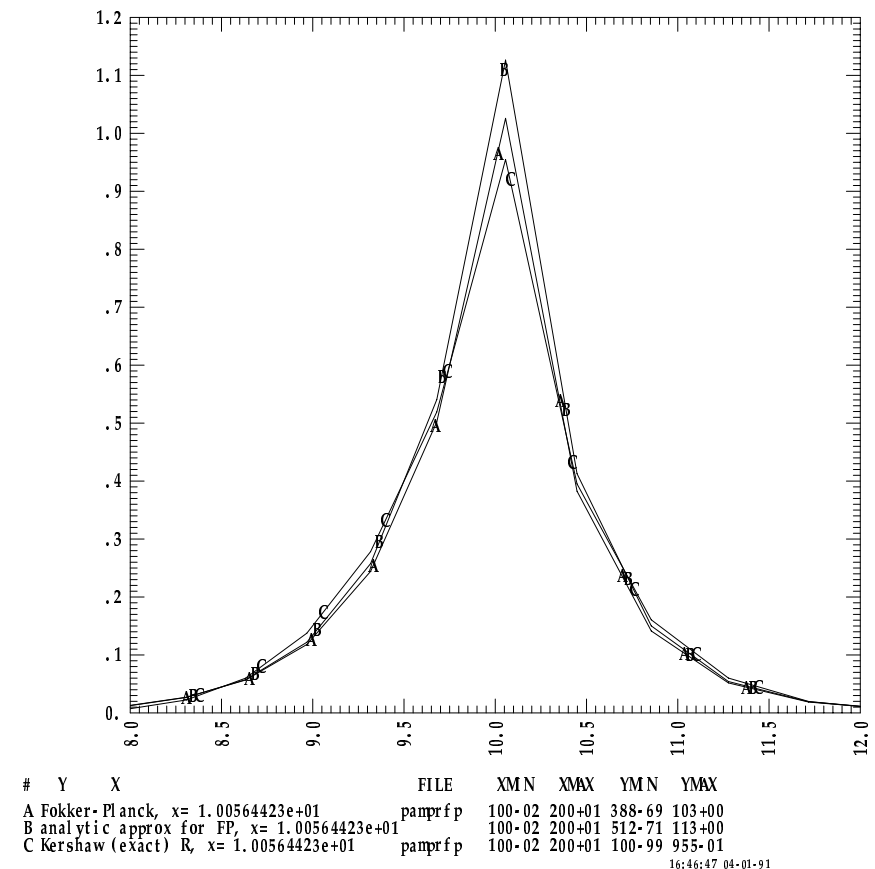
More sample calculations of $R-$ at large T and ν'

For $h\nu' \approx 100\,\text{keV}$, $kT = 50\,\text{keV}$.
A is the Green's function method,
B is an analytic approximation,
and C is exact.

Advantages of the Green's function method

- The approximate value of the scattering kernel can be found by inverting a tridiagonal matrix – a few FLOPS per matrix element

- The action of the kernel on a radiation spectrum can be found by solving a tridiagonal system, and the kernel need not be found

- The solution of an implicit equation for the spectrum is another tri-diagonal system

- The last two operations are much faster than any full matrix calculations — the same as Fokker-Planck solutions

- By using the Prasad, *et al.*, diffusion coefficient the results can be made more accurate for large T